The perturbed Bessel equation, I. A Duality Theorem.

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Abstract

The Euler-Gauss linear transformation formula for the hypergeometric function was extended by Goursat for the case of logarithmic singularities. By replacing the perturbed Bessel differential equation by a monodromic functional equation, and studying this equation separately from the differential equation by an appropriate Laplace-Borel technique, we associate with the latter equation another monodromic relation in the dual complex plane. This enables us to prove a duality theorem and to extend Goursat's formula to much larger classes of functions.

1 Introduction

We propose a new approach for the study of the perturbed Bessel differential equation (pBde). This approach is based on the duality between the functional monodromic relation generated by the pBde and its Laplace-Borel dual. The solutions of the latter relation can be viewed as generalized hypergeometric functions in the sense that they inherit a number of important properties of the classical hypergeometric functions.

The pBde, with an infinite number of parameters, gives rise to a monodromic relation with only two parameters

$$P\left(\zeta e^{\pi i}\right)=P\left(\zeta e^{-\pi i}\right)+Te^{-a\zeta}P\left(\zeta\right), \tag{1}$$

where $a>0, T$ is a complex constant which depends on all parameters of the pBde, $P\left(\zeta\right)$ is analytic at every point $\zeta\neq 0$ and bounded at infinity in any sectorial region. The dual function $F\left(t\right)$ is analytic in the t-plane cut along the interval $(-\infty,-a)$ of the real line and admits an analytical continuation along any path not containing points $t=0$ and $t=-a$. Among other properties which follow from (1) are that the function $F\left(t\right)$ has exponential growth at infinity of minimal type in any sectorial region, and satisfies the monodromic relation

$$F\left(\left(t+a\right)e^{\pi i}-a\right)=F\left(\left(t+a\right)e^{-\pi i}-a\right)-TF\left(-\left(t+a\right)e^{\pi i}\right). \tag{2}$$

The essence of our approach is in considering both the monodromic relations for the pBde and its dual quite separately from the differential equations which generate them. In brief, given a and T, we consider the linear spaces S and H of all functions retaining the analytic properties stated in the previous paragraph, and satisfying (1) and (2), respectively, and prove the following result:

Duality Theorem. *For fixed values of a and T, the Laplace transform operator $\mathcal{L} : H \to S$ is a bijection.*

The monodromic relation (2) does not, on its own, determine the behavior of $F(t)$ at $t = -a$. However, the relation (1), together with the conditions on $P(\zeta)$ stated above, shows that $F(t)$ has a logarithmic singularity at $t = -a$. Moreover we are able to prove the stronger statement:

Theorem 2. *For fixed values of a and T, assume that $F(t)$ belongs to the linear space H. Then in the region $|t + a| < a, |\arg(t + a)| < \pi$, the following relationship is valid*

$$F(t) = -TF(-(t+a))\log(t+a) + \sum_{k=0}^{\infty} A_k (t+a)^k, \tag{3}$$

where the series in (3) is absolutely convergent and A_k are complex coefficients which can be evaluated explicitly.

Formula (3) can be viewed as an extension of the special case of the Goursat linear transformation formula for the hypergeometric function ${}_2F_1(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}; t)$, valid when $\mathfrak{a} + \mathfrak{b} = \mathfrak{c} = 1$ and ${}_2F_1(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}; t)$ has a logarithmic singularity at $t = 1$. While Goursat's approach relied on properties specific to the hypergeometric function, we establish a linear transformation formula for all elements of space H.

The above results allow us, for fixed values of a and T, to provide all functions from the linear space S with asymptotic expansions in the ζ-plane, and using appropriate exponentially small terms to supply these expansions with uniform error bounds, see [5]. Returning to the original differential equation and developing our technique further, we derive an explicit formula for the connection coefficient T, see [6].

2 Two Relations for the Hypergeometric Function

In this section we consider the linear transformation formula and monodromic relation for the classical hypergeometric function ${}_2F_1(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}; t)$. The Euler linear transformation formula which determines the behavior of the hypergeometric function ${}_2F_1(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}; t)$ at point $t = 1$ is well known (see for example [1], **15.3.6**) and is valid for all values of the parameters except

for cases $\mathfrak{a}+\mathfrak{b}=\mathfrak{c}+m$, where m is an integer. The associated formula

$$\begin{aligned}&{}_2F_1\left(\mathfrak{a},\mathfrak{b};\mathfrak{c};1-(1-t)\,e^{\pm 2\pi i}\right)-\left({}_2F_1\left(\mathfrak{a},\mathfrak{b};\mathfrak{c};t\right)\right)\\ &=T^{\pm}(\mathfrak{a},\mathfrak{b},\mathfrak{c})\,(1-t)^{\mathfrak{c}-\mathfrak{a}-\mathfrak{b}}\,{}_2F_1\left(\mathfrak{c}-\mathfrak{a},\mathfrak{c}-\mathfrak{b};\mathfrak{c}-\mathfrak{a}-\mathfrak{b}+1;1-t\right),\end{aligned}\tag{4}$$

where

$$T^{\pm}(\mathfrak{a},\mathfrak{b},\mathfrak{c})=\mp 2\pi i e^{\pm\pi i(\mathfrak{c}-\mathfrak{a}-\mathfrak{b})}\frac{\Gamma\left(\mathfrak{c}\right)}{\Gamma\left(\mathfrak{a}\right)\Gamma\left(\mathfrak{b}\right)\Gamma\left(\mathfrak{c}-\mathfrak{a}-\mathfrak{b}+1\right)}\tag{5}$$

is less known and can be derived from the Euler linear transformation formula (see for example [7], section (4.2.4) formula (25). We refer to this formula as the *monodromic relation for* ${}_2F_1\left(\mathfrak{a},\mathfrak{b};\mathfrak{c};t\right)$. It is interesting to note that formula (4) is valid for all values of the parameters including cases $\mathfrak{a}+\mathfrak{b}=\mathfrak{c}+m$ referred to above. For the particular case $\mathfrak{a}+\mathfrak{b}=\mathfrak{c}$, this formula takes the form

$$\begin{aligned}&{}_2F_1\left(\mathfrak{a},\mathfrak{b};\mathfrak{a}+\mathfrak{b};1-(1-t)\,e^{\pm 2\pi i}\right)-{}_2F_1\left(\mathfrak{a},\mathfrak{b};\mathfrak{a}+\mathfrak{b};t\right)\\ &=\mp\, 2\pi i\frac{\Gamma\left(\mathfrak{a}+\mathfrak{b}\right)}{\Gamma\left(\mathfrak{a}\right)\Gamma\left(\mathfrak{b}\right)}{}_2F_1\left(\mathfrak{a},\mathfrak{b};1;1-t\right).\end{aligned}\tag{6}$$

In 1881, E. Goursat, [8], [9], when studying the logarithmic singularities of the hypergeometric function ${}_2F_1\left(\mathfrak{a},\mathfrak{b};\mathfrak{c};t\right)$, derived supplementary linear transformation formulas for the cases $\mathfrak{a}+\mathfrak{b}=\mathfrak{c}+m$ where m is an integer, which are quite different from Euler formula given in [1], **15.3.6**. In particular, he proved that if $\mathfrak{a}+\mathfrak{b}=\mathfrak{c}$ then for $|\arg(1-t)|<\pi$ and $|1-t|<1$ the following formula is valid, ([1], **15.3.10,**)

$$\begin{aligned}&2F_1\left(\mathfrak{a},\mathfrak{b};\mathfrak{a}+\mathfrak{b};t\right)\\ &=\frac{\Gamma\left(\mathfrak{a}+\mathfrak{b}\right)}{\Gamma\left(\mathfrak{a}\right)\Gamma\left(\mathfrak{b}\right)}\left(\sum_{n=0}^{\infty}\frac{\left(\mathfrak{a}\right)_n\left(\mathfrak{b}\right)_n}{(n!)^2}d(n)\,(1-t)^n-\log\left(1-t\right){}_2F_1\left(\mathfrak{a},\mathfrak{b};1;1-t\right)\right),\end{aligned}\tag{7}$$

where

$$d(n)=2\psi\left(n+1\right)-\psi\left(\mathfrak{a}+n\right)-\psi\left(\mathfrak{b}+n\right),\psi=\left(\log\Gamma\right)'.\tag{8}$$

Note that formula (7) also yields the monodromic relation (6). But the task of deriving (7) using only the monodromic relation (6) seems difficult, if not insurmountable, even for the particular case when $\mathfrak{a}+\mathfrak{b}=1$. In this latter case formula (6) takes the simpler form

$$\begin{aligned}&{}_2F_1\left(\mathfrak{a},1-\mathfrak{a};1;1-(1-t)\,e^{\pm 2\pi i}\right)-{}_2F_1\left(\mathfrak{a},1-\mathfrak{a};1;t\right)\\ &=T\left(\mathfrak{a}\right){}_2F_1\left(\mathfrak{a},1-\mathfrak{a};1;1-t\right),\end{aligned}\tag{9}$$

where

$$T(\mathfrak{a}) = \mp 2i \sin(\pi \mathfrak{a}), \tag{10}$$

and the same function occurs in each term of the relation. Even for this special case formula (7) remains as complicated as before.

Given $a > 0$, replacing $t \to -\frac{t}{a}$ in (9), this relation for the hypergeometric function ${}_2F_1\left(\mathfrak{a}, 1-\mathfrak{a}; 1; -\frac{t}{a}\right)$ can be rewritten in the form (2), where T is given by (10). In the next section we show that the dual equation (1) for (2) arises from the Bessel differential equation. It turns out that the monodromic relation generated by the pBde is indistinguishable from the analogous relation generated by the Bessel differential equation. Therefore by identifying additional minimal properties of ${}_2F_1(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}; t)$ that are required to derive (7) from (6) for the case $\mathfrak{a} + \mathfrak{b} = \mathfrak{c} = 1$, or Bessel's case, we find similar conditions for the monodromic relation for the pBde.

While our Theorem 2 appears to involve the derivation of (7) from (9), assuming $\mathfrak{a} + \mathfrak{b} = 1$, it in fact provides a partial answer to a deeper question regarding the additional properties of ${}_2F_1(\mathfrak{a}, \mathfrak{b}; \mathfrak{c}; t)$ required to derive (7) from (6). To answer this in full the results presented here for the pBde would need to be extended to a more general case of the perturbed Whittaker differential equation (pWde).

3 The Spaces $S_{a,T}$ and $H_{a,T}$

Given $a > 0$ and a complex number T, we define two function spaces $S_{a,T}$ and $H_{a,T}$ in terms of the monodromic relations (1) and (2).

3.1 Monodromic Relations generated by the pBde

We begin with a differential equation of the form

$$\frac{d^2u}{d\zeta^2} = \left(\frac{a^2}{4} + \frac{a_0}{\zeta^2} + \frac{a_1}{\zeta^4} + \dots\right) u, \tag{11}$$

where $a > 0$, $a_0, a_1, \dots$ are complex numbers, and the series is convergent for any complex $\zeta \neq 0$. Thus, if we set

$$A(\zeta) = a_0 + \frac{a_1}{\zeta^2} + \dots, \tag{12}$$

then $A\left(\zeta^{-1}\right)$ is an entire even function. Equation (11) can be considered as a perturbation of the standard Bessel equation. Indeed, upon applying the transformation $\zeta = iz, u = z^{\frac{1}{2}} y$, Bessel's equation

$$y'' + \frac{1}{z} y' + \left(1 - \frac{\nu^2}{z^2}\right) y = 0 \tag{13}$$

reduces to the following particular case of (11) with $a = 2$ and $A(\zeta) = \nu^2 - \frac{1}{4}$

$$\frac{d^2u}{d\zeta^2} = \left(1 - \frac{\frac{1}{4} - \nu^2}{\zeta^2}\right) u. \tag{14}$$

Clearly, (11) is invariant under the rotations $\zeta \to \zeta e^{\pm\pi i}$ and also under the reflection $\zeta \to -\zeta$. Since $\zeta = 0$ and $\zeta = \infty$ are the only singular points of (11), any solution of this differential equation admits an analytical continuation in the ζ-plane along any path not containing these points.

Let $u(\zeta)$ be a solution of (11) decaying on the positive ray. Representing this solution in the form

$$u(\zeta) = e^{-\frac{a}{2}\zeta} P(\zeta), \tag{15}$$

it is well known that the function $P(\zeta)$ has the same limit p_0 as $\zeta \to \infty$ along any ray in the region $-\frac{3\pi}{2} < \arg\zeta < \frac{3\pi}{2}$.

Any pair of solutions consisting of $u(\zeta)$ together with one of the functions $u(-\zeta), u\left(\zeta e^{\pi i}\right)$ and $u\left(\zeta e^{-\pi i}\right)$ forms a basis in the space of solutions of (11). So, for example, we have

$$u\left(\zeta e^{\pi i}\right) = Au\left(\zeta e^{-\pi i}\right) + Bu(\zeta), \tag{16}$$

where A and B are constants. Using (15) we can rewrite the relation (16) in the form

$$P\left(\zeta e^{\pi i}\right) = AP\left(\zeta e^{-\pi i}\right) + Be^{-a\zeta} P(\zeta). \tag{17}$$

Noting that all the functions $P\left(\zeta e^{-\pi i}\right), P(\zeta)$ and $P\left(\zeta e^{\pi i}\right)$ tend to p_0 as $\zeta \to +\infty$ and letting $\zeta \to +\infty$ in (17), it follows that $A = 1$. Setting $B = T$, we rewrite the relation (17) in the following final form

$$P\left(\zeta e^{\pi i}\right) = P\left(\zeta e^{-\pi i}\right) + Te^{-a\zeta} P(\zeta), \tag{18}$$

where

$$T = T(a, a_0, a_1, \ldots) \tag{19}$$

is a complex constant which is usually referred to as the *connection coefficient*, or *Stokes multiplier*. For the particular case of (11) given by (14) the constant T is known,

$$T = 2i\cos\nu\pi. \tag{20}$$

In the class of all differential equations with an irregular singular point at infinity of Poincaré rank 1 the pBde should be regarded as an exceptional or degenerate case in the sense that a pBde generates a pair of monodromic relations which are not intertwined (each relation involves only a single function) and the second relation can be derived from the first. For example, the relation for $P(-\zeta)$ follows immediately from (18).

Representing $P(\zeta)$ as a Laplace transform

$$P(\zeta) = \mathcal{L}\{F\} := \zeta \int_0^{+\infty} e^{-\zeta t} F(t)\, dt, \tag{21}$$

it can be shown using the standard Laplace-Borel technique that $F(t)$ satisfies the monodromic relation

$$F\left((t+a)\,e^{\pi i}-a\right)=F\left((t+a)\,e^{-\pi i}-a\right)-TF\left(-(t+a)\,e^{\pi i}\right). \qquad (22)$$

The relation (22) can be viewed as dual to (18). Moreover, it can be proved that $F(t)$ has all the other properties listed in the second paragraph of Section 1 and also satisfies relation (3). In particular, for Bessel's case given by (14) we have

$$F(t)={}_2F_1\left(\frac{1}{2}-\nu,\frac{1}{2}+\nu,1;-\frac{t}{2}\right), \qquad (23)$$

and as we noted in Section 2 the above properties are well known for this function. Analysis of the proof of the validity of these properties shows that instead of the pBde, we actually use the functional equation (18) and the above properties of $P(\zeta)$ which have been extracted from (11).

Considering the relation (18) separately from the differential equation (11), it turns out that (18), while being very much simpler than (11), yet remains a very rich source of information.

3.2 The linear space $S_{a,T}$

Given $a>0$ and complex T we introduce a linear space $S_{a,T}$ of functions $P(\zeta)$ satisfying the conditions:

i. $P(\zeta)$ is analytic in the ζ-plane punctured at the points $\zeta=0$ and $\zeta=\infty$;

ii. Given $P(\zeta)\in S_{a,T}$, there exists a positive decreasing function $M_P(r)$ such that if $|\arg\zeta|\le\pi$ and $|\zeta|\ge r$ for any $r, 0<r<\infty$, then

$$|P(\zeta)|\le M_P(r). \qquad (24)$$

iii. Every $P(\zeta)\in S_{a,T}$, is a solution of the equation (18).

Definition 1. *Given $P(\zeta)\in S_{a,T}$ we call the relation (18) the Stokes monodromic relation.*

If $P(\zeta)$ is given by (11) and (15) and T is given by (19) then $P(\zeta)\in S_{a,T}$. This shows that for any pair (a,T) there exists a non-trivial $P(\zeta)\in S_{a,T}$. For instance, given T with ν defined by (20), we represent a solution $u(\zeta)=u_\nu(\zeta)$ of the Bessel equation (14) decaying on the positive ray in the form $u_\nu(\zeta)=e^{-\zeta}P_\nu(\zeta)$. Clearly $P_\nu\left(\frac{a}{2}\zeta\right)\in S_{a,T}$.

Despite the fact that Definition 1 was suggested by a differential equation, in order to distinguish those techniques which depend on the equation itself from those which do not we have introduced a more general

object. The space $S_{a,T}$ with independent constants T and a may contain elements not associated with such a differential equation. Indeed, it is easy to check that given an entire function $\Phi(z)$ and $P(\zeta) \in S_{a,T}$ we have $\Phi\left(1/\zeta^2\right) P(\zeta) \in S_{a,T}$.

It follows from the definition of $S_{a,T}$ that any element $P(\zeta) \in S_{a,T}$ remains bounded at infinity in the region $|\arg \zeta| \leq \frac{3\pi}{2}$, and that in any sectorial region of the ζ-plane the function $P(\zeta)$ is of exponential growth at infinity with exponent a. More precisely, we claim that the following statements are valid.

Starting with $S_1 = 1$, $T_1 = T$ we introduce two sequences $\{S_k\}_{k=1}^{\infty}$ and $\{T_k\}_{k=1}^{\infty}$ which are determined by the following recurrence relations

$$S_{2m+1} = S_{2m}, \; S_{2m+2} = S_{2m+1} + T_{2m+1}T, \tag{25}$$

$$T_{2m+1} = T_{2m} + S_{2m}T, \; T_{2m+2} = T_{2m+1}. \tag{26}$$

Lemma 1. *Let $P(\zeta) \in S_{a,T}$ and let sequences $\{S_k\}_{k=1}^{\infty}$ and $\{T_k\}_{k=1}^{\infty}$ be defined by (25) and (26).Then for all integers $m \geq -1$ the following monodromic relations are valid*

$$P\left(\zeta e^{(2m+1)\pi i}\right) = S_{2m+1} P\left(\zeta e^{-\pi i}\right) + T_{2m+1} e^{-a\zeta} P(\zeta) \tag{27}$$

and

$$P\left(\zeta e^{(2m+2)\pi i}\right) = S_{2m+2} P(\zeta) + T_{2m+2} e^{a\zeta}\left(P\left(\zeta e^{-\pi i}\right)\right). \tag{28}$$

These relations can be extended to all negative values of k.

Illustration. One can check that for the Bessel case (14) the relations (27) and (28) with coefficients satisfying (25) and (26) are identical to the formulae for analytical continuations of Hankel's functions (4.13) and (4.14) from [2], respectively.

Proof. We will use mathematical induction on m to prove that the relations (27) and (28) are satisfied for some coefficients S_{2m+1}, T_{2m+1}, S_{2m+2} and T_{2m+2}, and in doing so we obtain the recurrence relations (27) and (28).

For $m = -1$ we have the trivial instance of (27), with $S_{-1} = 1$ and $T_{-1} = 0$; and the trivial instance of (28), with $S_0 = 1$ and $T_0 = 0$. For $m = 0$ (27) follows immediately from (18), along with the initial conditions $S_1 = 1$, $T_1 = T$. From (18) we also have, using (18),

$$P\left(\zeta e^{2\pi i}\right) = P\left(\left(\zeta e^{\pi i}\right) e^{\pi i}\right) = P\left(\left(\zeta e^{\pi i}\right) e^{-\pi i}\right) + T e^{-a\zeta e^{\pi i}} P\left(\zeta e^{\pi i}\right) = P(\zeta) + T e^{a\zeta} P\left(\zeta e^{\pi i}\right),$$

so that

$$P\left(\zeta e^{2\pi i}\right) = P(\zeta) + T e^{a\zeta} P\left(\zeta e^{\pi i}\right).$$

Now applying (18) again gives

$$P\left(\zeta e^{2\pi i}\right) = P(\zeta) + T e^{a\zeta}\left(P\left(\zeta e^{-\pi i}\right) + T e^{-a\zeta} P(\zeta)\right)$$

which yields finally

$$P\left(\zeta e^{2\pi i}\right)=\left(1+T^{2}\right)P\left(\zeta\right)+Te^{a\zeta}P\left(\zeta e^{-\pi i}\right).$$

This shows, incidentally, that $S_2=1+T^2$ and $T_2=T$, and so the recurrence relations hold when $m=0$.

Suppose now that odd and even hold for some value of m; we can use a similar process to prove that they also hold for $m+1$. Thus, using (28) we have

$$P\left(\zeta e^{(2(m+1)+1)\pi i}\right)=S_{2m+2}P\left(\zeta e^{\pi i}\right)+T_{2m+2}e^{a\zeta e^{\pi i}}P\left(\zeta\right)$$

Applying (18) gives

$$P\left(\zeta e^{(2m+3)\pi i}\right)=S_{2m+2}\left(P\left(\zeta e^{-\pi i}\right)+Te^{-a\zeta}P\left(\zeta\right)\right)+T_{2m+2}e^{-a\zeta}P\left(\zeta\right)$$

from which we obtain

$$P\left(\zeta e^{(2m+3)\pi i}\right)=S_{2m+2}P\left(\zeta e^{-\pi i}\right)+\left(T_{2m+2}+S_{2m+2}T\right)e^{-a\zeta}P\left(\zeta\right),$$

and also the corresponding recurrence relations.

In a similar way it can be shown that

$$P\left(\zeta e^{(2m+4)\pi i}\right)=\left(S_{2m+3}+T_{2m+3}T\right)P\left(\zeta\right)+T_{2m+3}e^{a\zeta}P\left(\zeta e^{-\pi i}\right),$$

and thus that the last pair of recurrence relations hold. ▲

The next statement is an immediate corollary of Lemma 1.

Lemma 2. *Let $P(\zeta)\in S_{a,T}$. Then for every sector*

$$S\left(r,\theta\right):=\left\{\zeta:|\zeta|\geq r,|\arg\zeta|\leq\theta\right\}$$

there exists a positive constant $M_P(\theta,r,T)$ such that for $\zeta\in S(r,\theta)$

$$\left|P\left(\zeta\right)\right|\leq M_P\left(\theta,r,T\right)\exp\left(a\left|\zeta\right|\right). \tag{29}$$

Proof. The validity of (29) follows from relations (27), (28) and the inequality (24).▲

We note that every $P(\zeta)\in S_{a,T}$ can be provided with an expansion of the form $\sum_{k=0}^{\infty}p_k/\zeta^k$, with complex coefficients p_k, valid in the region $-\frac{3\pi}{2}<\arg\zeta<\frac{3\pi}{2}$. Introducing the *remainders*

$$\mathrm{P}_n\left(\zeta\right):=P\left(\zeta\right)-\sum_{k=0}^{n-1}\frac{p_k}{\zeta^k},n=0,1,\ldots, \tag{30}$$

it is possible to prove that

Theorem 1. *For $P(\zeta)\in S_{a,T}$ the following estimates are valid for every $n=0,1,\ldots,$ and for $|\zeta|\geq r$*

- $$|\arg\zeta| \leq \frac{\pi}{2} \Rightarrow |\mathrm{P}_n(\zeta)| \leq M_P(r)\frac{n!}{a^n|\zeta|^n} \tag{31}$$

- $$|\arg\zeta| \leq \pi \Rightarrow |\mathrm{P}_n(\zeta)| \leq M_P(r)\frac{n!\sqrt{n+3}}{a^n|\zeta|^n}, \tag{32}$$

- $$\pi \leq |\arg\zeta| < \frac{3\pi}{2} \Rightarrow |\mathrm{P}_n(\zeta)| \leq 2M_P(r)\frac{n!\sqrt{n+3}}{a^n|\Re\{\zeta\}|^n}, \tag{33}$$

where $M_P(r)$ is a constant which is proportional to that from (24).

It follows immediately from (32) that for $n = 0, 1, \ldots$

$$|p_n| \leq M_P\frac{n!}{a^n}, M_P = \inf_{0<r<\infty} M_P(r). \tag{34}$$

The proof of Theorem 1 is based on the techniques of the paper [4], Section 5, using the integral representations (110) and (111) given there and Theorem 2 of the current paper. We will not prove this in detail here since a more general result will be presented in our paper [5].

The reason for the deterioration when passing from (32) to (33) is the appearance of exponentially small terms upon crossing the *Stokes rays* $\arg\zeta = \pm\pi$. This, which is a manifestation of the Stokes Phenomenon, will be discussed briefly in the next section.

We note finally that Lemma 1 and Lemma 2 allow us to extend the error bounds for $P(\zeta)$ given by (32) to the whole Riemann surface $-\infty < \arg\zeta < \infty$ by adding to the remainders appropriate exponentially small terms upon crossing the Stokes rays, $\arg\zeta = \pm m\pi$, $m \in \mathbb{N}$.

3.3 Exponentially small terms and Stokes' Phenomenon

The relation (18) provides an immediate clarification of the rôle of exponentially small terms in Stokes phenomenon, as we will now show.

Comparison of relations (32) and (33) shows that upon crossing the rays $\arg\zeta = \pm\pi$ the approximation given by (32) begins to deteriorate and the greater the deviation from the ray the greater this deterioration becomes.

The cause of the above deterioration is hidden in relation (18). Indeed, relation (18) generates the system of relations

$$\mathrm{P}_n\left(\zeta e^{\pi i}\right) = \mathrm{P}_n\left(\zeta e^{-\pi i}\right) + Te^{-a\zeta}P(\zeta), \tag{35}$$

where $n = 0, 1, \ldots$ and $\mathrm{P}_0(\zeta) = P(\zeta)$. Let us introduce functions

$$E^+(\zeta) = Te^{a\zeta}P\left(\zeta e^{-\pi i}\right), \pi \leq \arg\zeta < \frac{3\pi}{2} \tag{36}$$

$$E^-(\zeta) = -Te^{a\zeta}P\left(\zeta e^{\pi i}\right), -\frac{3\pi}{2} < \arg\zeta \leq -\pi. \tag{37}$$

These functions decay exponentially in the regions $\left\{\zeta:\pi\leq\arg\zeta<\frac{3\pi}{2}\right\}$ and $\left\{\zeta:-\frac{3\pi}{2}<\arg\zeta\leq-\pi\right\}$, respectively, as $\zeta\to\infty$. Using (35) we observe that subtracting these exponentially small terms from the remainders, we retrieve the accuracy of (32):

$$\pi\leq\arg\zeta\leq\frac{3\pi}{2}\Longrightarrow\left|\mathrm{P}_n\left(\zeta\right)-E^+\left(\zeta\right)\right|\leq M_P\left(r\right)\frac{\sqrt{n+3}n!}{a^n|\zeta^n|},\qquad(38)$$

$$-\frac{3\pi}{2}\leq\arg\zeta\leq-\pi\Longrightarrow\left|\mathrm{P}_n\left(\zeta\right)-E^-\left(\zeta\right)\right|\leq M_P\left(r\right)\frac{\sqrt{n+3}n!}{a^n|\zeta^n|},\qquad(39)$$

Indeed, it follows from (35), (36) and (37) that for $n=0,1,\ldots$

$$\pi\leq\arg\zeta\leq\frac{3\pi}{2}\Longrightarrow\arg\left(\zeta e^{-2\pi i}\right)\in[-\pi,\pi]:\mathrm{P}_n\left(\zeta\right)-E^+\left(\zeta\right)=\mathrm{P}_n\left(\zeta e^{-2\pi i}\right),$$

$$-\frac{3\pi}{2}<\arg\zeta\leq-\pi\Longrightarrow\arg\left(\zeta e^{2\pi i}\right)\in[-\pi,\pi]:\mathrm{P}_n\left(\zeta\right)-E^-\left(\zeta\right)=\mathrm{P}_n\left(\zeta e^{2\pi i}\right),$$

which, using (32) yields (38) and (39).

Note that if the monodromic relation is generated by the differential equation (11) then evaluation of the exponentially small terms reduces to evaluation of the constant T. To clarify the cause for the above phenomenon we must turn to the dual complex plane and to consider a dual space $H_{a,T}$ of functions $F\left(t\right)$ analytic in the dual plane.

3.4 The linear space $H_{a,T}$

Given $a>0$ and complex T we introduce a second linear space $H_{a,T}$ of functions $F\left(t\right)$ satisfying the conditions:

- (i) $F\left(t\right)$ is an analytic function in the complex t-plane punctured at the two finite points $t=0$ and $t=-a$, and with exponential growth of minimal type at ∞ in any sectorial region $\{t:-\infty<\alpha<\arg t<\beta<+\infty\}$;
- (ii) there exists a branch of $F(t)$ which is analytic in the t-plane cut along $(-\infty,-a]$;
- (iii) the branch given by (ii) satisfies the estimate $F(t)=O\left(\log\left(t+a\right)\right)$ as $t\to-a$, $|t+a|<a$, $|\arg\left(t+a\right)|<\pi$;
- (iv) the branch given by (ii) satisfies the monodromic relation (22) given by

  $$F\left(\left(t+a\right)e^{\pi i}-a\right)=F\left(\left(t+a\right)e^{-\pi i}-a\right)-TF\left(-\left(t+a\right)e^{\pi i}\right),\quad(40)$$

  where $-a<t<+\infty$, and T is the constant from (18).

Note that the points $(t+a)\,e^{\pi i}-a$ and $(t+a)\,e^{-\pi i}-a$ belong to the upper and lower banks of the cut, respectively, and the relation (40) can be rewritten as

$$F\left(te^{\pi i}-a\right)=F\left(te^{-\pi i}-a\right)-TF\left(-te^{\pi i}\right),t>0,$$

as well as in the equivalent form

$$F\left((t+a)\,e^{-2\pi i}-a\right)=F\,(t)+TF\,(-\,(t+a))\,, \tag{41}$$

where t belongs to the upper bank of the cut.

Definition 2. *Given $F(t)\in H_{a,T}$ we call the relation (40) the hypergeometric monodromic relation.*

Remark 1. *Since function $F(t)$ is multi-valued, it is necessary to interpret the expression $F(-t)$ "carefully". If $F_0(t)$ is a branch of $F(t)$ which is given by (ii) of Definition 2, then $F_0(-t)$ is a single-valued analytic function in the t-plane cut along $[a,+\infty)$ such that*

$$F_0(-t)=\begin{cases}F_0\left(te^{\pi i}\right) & \text{if } \Im t<0\,,\\ F_0\left(te^{-\pi i}\right) & \text{if } \Im t>0\,.\end{cases}$$

We define $F(-t)$ as an analytical continuation of $F_0(-t)$ to the complex t-plane punctured at the two points $t=0$ and $t=-a$. The function $F(-t-a)$ satisfies condition (i).

Below we will also use $F(t)$ to denote the branch given by condition (ii), where this will cause no confusion.

For the case given by (23) setting $F_\nu(t)={}_2F_1\left(\frac{1}{2}-\nu,\frac{1}{2}+\nu,1;-\frac{t}{a}\right)$, where $a=\frac{1}{2}-\nu$ and $T=2i\cos\nu\pi$, one can check that $F_\nu(t)\in H_{a,T}$.

In the next section we will show that there is a one-to-one correspondence between the linear spaces $H_{a,T}$ and $S_{a,T}$.

4 The Duality theorem

Defining the Laplace transform operator $\mathcal{L}$ by (21) we claim that

Theorem 2. *Given a and T, the operator $\mathcal{L}:H_{a,T}\to S_{a,T}$ is a bijection.*

We present the proof in two parts.

4.1 Part I: $\mathcal{L}H_{a,T}\subset S_{a,T}$

Proof. Let $F(t)\in H_{a,T}$ and let $P(\zeta)$ be given by (21). Then it follows from conditions (i) and (ii) of section 2.3 that $P(\zeta)$ admits an analytical continuation from the right half-plane to the ζ-plane cut along the interval $(-\infty,0)$ as a function bounded at ∞ and continuous in the cut plane except

for $\zeta = 0$. Moreover, for $-\pi \leq \theta \leq \pi$ and for $0 < \rho < +\infty$, using Cauchy's theorem, the integral representation

$$P\left(\rho e^{i\theta}\right) = \rho e^{i\theta} \int_0^{\infty \cdot e^{-i\theta}} e^{-\rho e^{i\theta} t} F(t)\, dt \tag{42}$$

is valid, where the integral is absolutely convergent. Setting $\theta = \pm\pi$ in (42), we retain the previous convergence due to the condition (iii) of 2.3. Subtracting the second integral from the first we have

$$P\left(\rho e^{\pi i}\right) - P\left(\rho e^{-\pi i}\right) = -\rho \int_0^{\infty \cdot e^{-\pi i}} e^{\rho t} F(t)\, dt + \rho \int_0^{\infty \cdot e^{\pi i}} e^{\rho t} F(t)\, dt. \tag{43}$$

Due to condition (ii) $F(t)$ is analytic in the circle $|t| < a$, and thus for $t \in (-a, 0)$ we have $F(t) = F(t^*)$, where $t^* = (t+a)e^{-2\pi i} - a$. Therefore, writing $\int_{-a}^{-\infty}$ for an integral along the upper bank of the cut, (43) can be rewritten in the form

$$P\left(\rho e^{\pi i}\right) - P\left(\rho e^{-\pi i}\right) = -\rho \int_{-a}^{-\infty} e^{\rho t} F(t^*)\, dt + \rho \int_{-a}^{-\infty} e^{\rho t} F(t)\, dt \tag{44}$$

$$= -\rho \int_{-a}^{-\infty} e^{\rho t} \left(F(t^*) - F(t)\right) dt. \tag{45}$$

Using the monodromic relation (40) in the form (41), the last integral in (45) can be rewritten as

$$-T\rho \int_{-a}^{-\infty} e^{\rho t} F(-(t+a))\, dt = T\rho e^{-a\rho} \int_{-\infty}^{-a} e^{\rho(t+a)} F(-(t+a))\, dt. \tag{46}$$

Finally we have

$$T\rho e^{-a\rho} \int_{-\infty}^{0} e^{\rho t} F(-t)\, dt = T\rho e^{-a\rho} \int_0^{+\infty} e^{-\rho t} F(t)\, dt = Te^{-a\rho} P(\rho). \tag{47}$$

The chain of relations (45)–(47) shows that $P(\zeta)$ satisfies the condition (iii) of definition 1, and thus, $P(\zeta) \in S_{a,T}$.▲

4.2 II. $S_{a,T} \subset \mathcal{L}H_{a,T}$

We assume now that $P(\zeta) \in S_{a,T}$. We set $F(t)$ equal to the Borel transform of $P(\zeta)$, so that $P(\zeta) = \mathcal{L}\{F\}$, and prove, in five steps, that $F(t) \in H_{a,T}$.

(1) Integral representation for $F(t)$ for $|\arg t| < \frac{\pi}{2}$. Given $r > 0$ and $\theta > 0$, we introduce the contour $\gamma_\theta(r)$ with an anti-clockwise orientation as follows:

$$\gamma_\theta(r) = l_{-\theta}(r) \cup C_\theta(r) \cup l_\theta(r), \tag{48}$$

where

$$l_\theta(r) = \left\{\zeta : \zeta = \rho e^{i\theta}, r < \rho < \infty\right\} \tag{49}$$

and

$$C_\theta(r) = \{\zeta : -\theta < \arg\zeta < \theta, |\zeta| = r\}. \tag{50}$$

For $t > 0$ the function $F(t)$ can then be represented in the form

$$F(t) = \frac{1}{2\pi i}\int_{\gamma_\theta(r)} e^{t\zeta} P(\zeta)\frac{d\zeta}{\zeta}, \tag{51}$$

with any $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $r > 0$.

Below we set $\theta = \pi$ and omit the subscript π in $C_\pi(r)$ and in $\gamma_\pi(r)$ if this does not lead to confusion.

First we prove that this integral converges for $t > 0$. Then we demonstrate how, by changing the integral representation, we can obtain an analytical continuation of $F(t)$ that allows us to check the validity of conditions (i)-(iii) of 2.3. Finally we show that the function $F(t)$ satisfies the relation (40). Setting

$$F^+(t,r) = \frac{1}{2\pi i}\int_{l_\pi(r)} e^{t\zeta} P(\zeta)\frac{d\zeta}{\zeta}, \tag{52}$$

$$F^-(t,r) = \frac{1}{2\pi i}\int_{l_{-\pi}(r)} e^{t\zeta} P(\zeta)\frac{d\zeta}{\zeta} \tag{53}$$

and

$$F_0(t,r) = \frac{1}{2\pi i}\int_{C(r)} e^{t\zeta} P(\zeta)\frac{d\zeta}{\zeta} \tag{54}$$

we have

$$F(t) = F^+(t,r) - F^-(t,r) + F_0(t,r). \tag{55}$$

Observe, that $F_0(t,r)$ admits an analytical continuation from the positive ray to the whole t-plane as an entire function of t. Since $P(\zeta) \in S_{a,T}$, the relation (24) yields the following estimate valid in the whole t-plane

$$|F_0(t,r)| \le M_P(r)\, e^{r|t|} \tag{56}$$

meaning that $F_0(t,r)$ is an entire function of exponential type r. On the other hand, for the functions $F^\pm(t,r)$ given by (52) and (53) the inequality (24) yields the estimates

$$\left|F^\pm(t,r)\right| \le \frac{M_P(r)}{2\pi r}\int_r^{+\infty} e^{-\zeta t} d\zeta = \frac{M_P(r)}{2\pi r} e^{-rt}, 0 < t < +\infty. \tag{57}$$

The estimates (56) and (57) show that integrals (52) – (54) are absolutely convergent, so that the function $F(t)$ is well defined for positive t, and that its growth on the positive ray is determined essentially by estimate (56). It follows from (56) and the preceding comments that the integral in (21) is absolutely convergent for $\Re\{\zeta\} > r$. Since r is an arbitrarily small positive number it follows that this integral is absolutely convergent for all ζ such

that $|\arg \zeta| < \frac{\pi}{2}$. Therefore formula (51) can be considered as the converse for the Laplace transform (21), which proves the validity of the representation

$$P(\zeta) = \zeta \int_0^\infty e^{-s\zeta} F(s)\, ds, -\frac{\pi}{2} < \arg \zeta < \frac{\pi}{2}. \tag{58}$$

(2) Representation for $F(t)$ in the region $\{t : |t| > a\}$. Let θ be any angle, $-\infty < \theta < \infty$. To obtain the desired analytical continuation of $F(t)$ we rotate the path of integration in (51), with $\theta = \pi$, about the origin through an angle $-\theta$ while simultaneously increasing $\arg t$ by θ. Lemma 1 yields

$$|P(\zeta)| \leq M(\theta, r, T) \exp(a|\zeta|), \tag{59}$$

where the positive parameter $M(\theta, r, T)$ depends on θ, r and T. Following the above rotation the integral in the form

$$F\left(te^{i\theta}\right) := \frac{1}{2\pi i} \int_{\gamma(r)\cdot e^{-i\theta}} e^{te^{i\theta}\zeta} P(\zeta) \frac{d\zeta}{\zeta}, 0 < t < \infty, \tag{60}$$

retains the absolute convergence of the integral (51). Thus, using Cauchy's theorem, it can be shown that (60) provides an analytical continuation of (51). Since θ is arbitrary, this shows that $F(t)$ admits an analytical continuation to the exterior of the circle $\{t : |t| = a\}$, and (60) provides an integral representation for $F(t)$ in the region $\{t : |\arg t| < \pi\}$ and also in the region $\{t : |t| > a, -\infty < \arg t < \infty\}$. It also follows from (60) and (59) that given $0 < r < \infty$ in any sectorial region $\{t : |t| \geq R > a, \alpha < \arg t < \beta\}$, of the t-plane the following estimate is valid

$$|F(t)| \leq Ke^{r|t|}, \tag{61}$$

where the positive parameter K depends on R, α, β, and r.

(3) Representation of $F(t)$ in the region $\left(t : |\arg(t+a)| < \frac{\pi}{2}\right)$. Next we prove that $F(t)$ is analytic in the half-plane $\Re z > -a$ and therefore it satisfies the conditions (i) and (ii) of 2.3. In order to prove this we first assume that $0 < t < \infty$ and derive another integral representation of $F(t)$. Now, returning to the representation (51) and making, in (52) and (53), the change of variable suggested by (49) for $\theta = \pm\pi$, it follows that (51) can be written in the form

$$F(t) = F_0(t, r) + \frac{1}{2\pi i} \int_r^{+\infty} e^{-t\zeta} \left(P\left(\zeta e^{\pi i}\right) - P\left(\zeta e^{-\pi i}\right)\right) \frac{d\zeta}{\zeta}, \tag{62}$$

where $F_0(t, r)$ is given by (54).

Using the relation

$$P\left(\zeta e^{i\pi}\right) - P\left(\zeta e^{-i\pi}\right) = Te^{-a\zeta} P(\zeta),$$

which follows from (18), we can represent $F(t)$, $t>0$ in the form

$$F(t)=F_0(t,r)+\frac{T}{2\pi i}\int_r^{+\infty}e^{-(t+a)\zeta}P(\zeta)\frac{d\zeta}{\zeta}. \tag{63}$$

Formula (63) shows that $F(t)$ can be continued analytically from the positive ray to the half-plane $|\arg(t+a)|<\frac{\pi}{2}$. This completes the proof of step **(3).**

Steps **(1)-(3)** prove that $F(t)$ is analytic in the whole t-plane punctured at $t=0$ and $t=-a$, and formula (63) represents a branch of $F(t)$ which is analytic in the plane cut along the interval $(-\infty,-a]$. We keep the notation $F(t)$ for this branch, if it does not lead to confusion. Clearly the function $F(t)$ is analytic in the circle $|t|<a$. Applying Watson's lemma to the relation (58) we can provide every element $P(\zeta)\in S_{a,T}$ with an asymptotic expansion of the form $\sum_{k=0}^{\infty}p_k/\zeta^k$, where

$$p_k=F^{(k)}(0),k=0,1,\ldots,F=\mathcal{L}^{-1}P. \tag{64}$$

Analytical properties of $F(t)$ given by (1)-(3) show that the relation

$$P(\zeta)-\sum_{k=0}^{n-1}p_k/\zeta^k=O\left(\frac{1}{\zeta^n}\right),\zeta\to\infty, \tag{65}$$

is valid for every $n\in\mathbb{N}$ and for ζ satisfying $-\frac{\pi}{2}<\arg\zeta<\frac{\pi}{2}$, and can be extended to every sub-sector of the region $-\frac{3\pi}{2}<\arg\zeta<\frac{3\pi}{2}$. Next we use this fact to prove that $F(t)$ satisfies condition (iii) of 2.3.

(4) Behavior of $F(t)$ at $t=-a$. We prove the following statement.

Lemma 2. *Let $P(\zeta)\in S_{a,T}$ and $F(t)$ be given by (63), then there exists a constant A_0 such that in the region $|t+a|<a,|\arg(t+a)|<\pi$ the following formula is valid*

$$F(t)=-\frac{Tp_0}{2\pi i}\log(t+a)+A_0+o(1),t\to-a. \tag{66}$$

Proof. To prove (66) we use (63) to estimate $F(t)$. Note that $F_0(t,r)$ is an entire function in the t-plane and from (54) we may write

$$F_0(t,r)=\frac{1}{2\pi i}\int\limits_{C(r)}e^{(t+a)\zeta}e^{-a\zeta}P(\zeta)\frac{d\zeta}{\zeta}.$$

Since $e^{(t+a)\zeta}=1+O(t+a)$ as $t\to-a$ it then follows that

$$F_0(t,r)=\frac{1}{2\pi i}\int\limits_{C(r)}e^{-a\zeta}P(\zeta)\frac{d\zeta}{\zeta}+O(t+a),t\to-a. \tag{67}$$

Let us denote the integral on the right hand side of (63) by $I(t,r)$ and observe that it can be written in the form

$$I(t,r)=\frac{T}{2\pi i}\int_r^{+\infty}e^{-(t+a)\zeta}\frac{P(\zeta)-p_0}{\zeta}d\zeta+\frac{Tp_0}{2\pi i}\int_r^{+\infty}e^{-(t+a)\zeta}\frac{d\zeta}{\zeta}, \tag{68}$$

where p_0 is given by (64) with $k = 0$. Using the relation (65) with $n = 1$ together with $e^{(t+a)\zeta} = 1 + O(t+a)$ as $t \to -a$, the first integral in (68) can be represented as

$$\frac{T}{2\pi i}\int_r^{+\infty} e^{-(t+a)\zeta}\frac{P(\zeta)-p_0}{\zeta}d\zeta = \frac{T}{2\pi i}\int_r^{+\infty}\frac{P(\zeta)-p_0}{\zeta}d\zeta + O(t+a), t \to -a, \tag{69}$$

where the integral on the right hand-side is absolutely convergent.

Finally we evaluate the second integral in (68) using the exponential integral

$$E_1(z) = \int_1^{\infty} e^{-z\tau}\frac{d\tau}{\tau} \tag{70}$$

and its power series expansion, see formula (**5.1.11**) in [1],

$$E_1(z) = -\log z - \gamma - \sum_{m=1}^{\infty}\frac{(-z)^m}{(m+1)\,m!}, \tag{71}$$

where $|\arg z| < \pi$ and γ is Euler's constant. Thus, replacing z in (70) by $r(t+a)$, and applying (71) we have, for $|\arg(t+a)| < \pi$,

$$\begin{aligned}\frac{Tp_0}{2\pi i}\int_r^{+\infty} e^{-(t+a)\zeta}\frac{d\zeta}{\zeta} &= \frac{Tp_0}{2\pi i}E_1(r(t+a))\\ =&\frac{Tp_0}{2\pi i}\left(-\log(r(t+a)) - \gamma - \sum_{m=1}^{\infty}\frac{(-r(t+a))^m}{(m+1)\,m!}\right).\end{aligned} \tag{72}$$

Combining (67), (69) and (72), we obtain the following estimate for $F(t)$

$$F(t) = -\frac{TF(0)}{2\pi i}\log(t+a) + A_0 + O(t+a), t \to -a, |\arg(t+a)| < \pi \tag{73}$$

where

$$A_0 = \frac{1}{2\pi i}\int\limits_{C(r)} e^{-a\zeta}P(\zeta)\frac{d\zeta}{\zeta} + \frac{T}{2\pi i}\int_r^{+\infty}\frac{P(\zeta)-p_0}{\zeta}d\zeta - \frac{Tp_0}{2\pi i}(\gamma + \log r), \tag{74}$$

and $r \in (0,\infty)$. While the individual terms on the righthand side of (74) are clearly dependent on r it follows from (73), taking the limit as $t \to -a$,that the coefficient A_0 does not depend on r. ▲

It remains to be proven that $F(t)$ satisfies the monodromic relation (40).

(5) Derivation of the monodromic relation for $F(t)$. The properties of $F(t)$ given by (i) and (ii) of the section 2.3. which have been proven in the preceding paragraph allow us to use the relations (43) and (45) and to show that for $0 < \rho < +\infty$

$$P\left(\rho e^{\pi i}\right) - P\left(\rho e^{-\pi i}\right) = \rho\int_{-a}^{-\infty} e^{\rho t}\left(F(t^*) - F(t)\right)dt,$$

where $t^* = (t+a)\,e^{-2\pi i} - a$. On the other hand using elementary transformations given by (46) and (47) we have

$$Te^{-\rho t}P(\rho) = T\rho\int_{-a}^{-\infty} e^{\rho t}F(-(t+a))\,dt.$$

Since the $P\left(\rho e^{\pi i}\right) - P\left(\rho e^{-\pi i}\right) = Te^{-\rho t}P(\rho)$, it follows that $F(t^*) - F(t) = TF(-(t+a))$.

Steps (**1**)-(**5**) justify the inclusion $S_{a,T} \subset \mathcal{L}H_{a,T}$. ▲

In the next section we use the Duality theorem to prove a version of the linear transformation formula for $F(t) \in H_{a,T}$.

5 Linear transformation formula

The following statements is valid.

Theorem 3. *Assume that $F(t) \in H_{a,T}$. Then for the branch of $F(t)$ given by condition (ii) of 2.3 in the region $|t+a| < a, |\arg(t+a)| < \pi$ the following formula is valid*

$$F(t) = \sum_{k=0}^{\infty} A_k\,(t+a)^k - \frac{T}{2\pi i}F(-(t+a))\log(t+a), \qquad (75)$$

where the series in (75) is absolutely convergent, A_0 is given by (74), where $p_0 = F(0)$, and A_k, $k \in \mathbb{N}$, are complex coefficients which can be found explicitly.

Remark 2. *For Bessel's case given by (14) relation (75) follows immediately from a degenerate case of the Euler transformation formula for the hypergeometric function, see* 15.3.10 *of [1] and Appendix 1. Thus, the relation (75) can be viewed as a generalization for the elements of $H_{a,T}$ of the linear transformation formula (7) with $\mathfrak{a} + \mathfrak{b} = 1$.*

Proof of Theorem 3. The relation (66) of Lemma 2 can be rewritten as

$$F(t) = -\frac{TF(0)}{2\pi i}\log(t+a) + A_0 + o(1), t \to -a, \qquad (76)$$

where A_0 is given by (74). Since $F(t+a)$ is analytic in the circle $|t+a| < a$ we have the relation

$$F(-(t+a))\log(t+a) = (F(0) + O(t+a))\log(t+a) \text{ as } t \to -a,$$

which yields

$$F(0)\log(t+a) = F(-(t+a))\log(t+a) + o(1).$$

Substituting the last relation into (76) allows us to rewrite it as

$$F(t) = -\frac{T}{2\pi i}F(-(t+a))\log(t+a) + A_0 + o(1), t \to -a, \qquad (77)$$

Let us introduce a function $\Phi(t)$ such that

$$F(t) = -\frac{T}{2\pi i} F(-(t+a)) \log(t+a) + A_0 + \Phi(t+a), t \to -a. \tag{78}$$

Clearly $\Phi(t)$ is analytic in the t-plane punctured at the two points $t = 0$ and $t = a$. On the other hand the relation (77) shows that

$$\Phi(t+a) = o(1), t \to -a. \tag{79}$$

Since $|t+a| < a, |\arg(t+a)| < \pi$ we can continue analytically both sides of (78) by rotating $t+a$ through angles π and $-\pi$ about the origin. This allows us to rewrite (78) as

$$\begin{aligned} F\left((t+a)e^{\pi i} - a\right) = A_0 + \Phi\left((t+a)e^{\pi i}\right) \\ +\frac{T}{2\pi i} F\left(-(t+a)e^{\pi i}\right) \log\left((t+a)e^{\pi i}\right), \end{aligned} \tag{80}$$

and also as

$$\begin{aligned} F\left((t+a)e^{-\pi i} - a\right) = A_0 + \Phi\left((t+a)e^{-\pi i}\right) \\ -\frac{T}{2\pi i} F\left(-(t+a)e^{-\pi i}\right) \log\left((t+a)e^{-\pi i}\right). \end{aligned} \tag{81}$$

Subtracting (81)from (80) and using the monodromic relation (41) gives

$$F\left((t+a)e^{\pi i} - a\right) - F\left((t+a)e^{-\pi i} - a\right) = -TF\left(-(t+a)e^{\pi i}\right).$$

A straightforward calculation using

$$F\left(-(t+a)e^{\pi i}\right) \equiv F\left(-(t+a)e^{-\pi i}\right), |t+a| < a,$$

shows that

$$-TF\left(-(t+a)e^{\pi i}\right) = -\frac{T}{2\pi i} F\left(-(t+a)e^{\pi i}\right) \log\left(e^{2\pi i}\right) + \Phi\left((t+a)e^{\pi i}\right) - \Phi\left((t+a)e^{-\pi i}\right).$$

Canceling equal terms, the last relation can be simplified and we have

$$\Phi\left((t+a)e^{\pi i}\right) = \Phi\left((t+a)e^{-\pi i}\right)$$

which, using (79), shows that $\Phi(t)$ is analytic and single-valued in the circle $|t+a| < a$ and that $\Phi(0) = 0$. Thus,

$$\Phi(t+a) = \sum_{k=1}^{\infty} A_k (t+a)^k, A_k = \frac{\Phi^{(k)}(-a)}{k!}.$$

▲

6 The coefficients A_k

In this section we describe a procedure that allows us to derive formulas for computing the coefficients $A_k, k = 1, \dots,$ given by (75). Simultaneously we give an alternative proof of Theorem 3 which is based on Theorem 1. Returning to the relation (63) rewritten as

$$F(t) - F_0(t, r) = \frac{T}{2\pi i} \int_r^{+\infty} e^{-(t+a)\zeta} P(\zeta) \frac{d\zeta}{\zeta}, \tag{82}$$

where $F_0(t, r)$ is given by (54) we represent the integral in (82) in the form

$$\frac{T}{2\pi i} \int_r^{+\infty} e^{-(t+a)\zeta} P(\zeta) \frac{d\zeta}{\zeta} = I_n(t, r) + J_n(t, r) \tag{83}$$

where

$$I_n(t, r) = \frac{T}{2\pi i} \int_r^{+\infty} e^{-(t+a)\zeta} \left(P(\zeta) - \sum_{k=0}^{n} p_k/\zeta^k \right) \frac{d\zeta}{\zeta} \tag{84}$$

and

$$J_n(t, r) = \frac{T}{2\pi i} \sum_{k=0}^{n} p_k \int_r^{+\infty} e^{-(t+a)\zeta} \frac{d\zeta}{\zeta^{k+1}}. \tag{85}$$

Given $n \in \mathbb{N}$, and expanding $I_n(t, r)$, $J_n(t, r)$, and $F_0(t, r)$ in series of the form

$$\log(t+a) \sum_k \beta_k (t+a)^k + \sum_k \alpha_k (t+a)^k,$$

we are only interested in the values of the coefficients α_n in each series. Let us denote these coefficients by $\alpha_n(I)$, $\alpha_n(J)$ and $\alpha_n(F_0)$ for $I_n(t, r)$, $J_n(t, r)$ and $F_0(t, r)$, respectively. Then (82) shows that the coefficient A_n at $(t+a)^n$ can be written as

$$A_n = \alpha_n(I) + \alpha_n(J) + \alpha_n(F_0). \tag{86}$$

Firstly, expanding the exponential $e^{-(t+a)\zeta}$ in (84) into a Taylor series we have

$$I_n(t, r) = \frac{T}{2\pi i} \int_r^{+\infty} \sum_{m=0}^{n} \frac{(-1)^m (t+a)^m \zeta^m}{m!} \left(P(\zeta) - \sum_{k=0}^{n} p_k/\zeta^k \right) \frac{d\zeta}{\zeta} + o((t+a)^n), t \to -a, \tag{87}$$

so that

$$\alpha_n(I) = (-1)^n \frac{T}{2\pi i} \frac{1}{n!} \int_r^{+\infty} \left(P(\zeta) - \sum_{k=0}^{n} p_k/\zeta^k \right) \zeta^{n-1} d\zeta, \tag{88}$$

and the integral is absolutely convergent since $\left(P(\zeta)-\sum_{k=0}^{n}p_k/\zeta^k\right)\zeta^{n-1}=O\left(\frac{1}{\zeta^2}\right)$ as $\zeta\to\infty$. Moreover, using error bound (32) of Theorem 1 yields the estimate for $n\in\mathbb{N}$

$$|\alpha_n(I)|\le\frac{|T|\,M_P}{2\pi}\frac{\sqrt{n+3}}{a^n},M_P=\inf_{0<r<\infty}M_P(r). \tag{89}$$

Secondly, we calculate $J_n(t,r)$ using the exponential integral

$$E_n(z)=\int_1^\infty e^{-zt}\frac{dt}{t^n},n=1,2,\ldots, \tag{90}$$

and its power series expansion

$$E_n(z)=\frac{(-z)^{n-1}}{(n-1)!}(-\log z+\psi(n))-\sum_{m=0,m\ne n-1}^{\infty}\frac{(-z)^m}{(m-n+1)\,m!}, \tag{91}$$

where

$$|\arg z|<\pi,\psi(1)=-\gamma,\psi(n)=-\gamma+\sum_{m=1}^{n-1}\frac{1}{m},$$

and γ is Euler's constant.

We have

$$J_n(t,r)=\frac{T}{2\pi i}\sum_{k=0}^{n}\frac{p_k}{r^k}E_{k+1}(r(t+a)) \tag{92}$$

and substituting (91) into (92) yields

$$\begin{aligned}J_n(t,r)\quad&=\quad\frac{T}{2\pi i}\sum_{k=0}^{n}p_k\frac{(-(t+a))^k}{k!}(-\log r-\log(t+a)+\psi(k+1))\\&\quad-\frac{T}{2\pi i}\sum_{k=0}^{n}\frac{p_k}{r^k}\sum_{m=0,m\ne k}^{\infty}\frac{(-r(t+a))^m}{(m-k)\,m!}.\end{aligned} \tag{93}$$

It follows from (93) that the coefficient $\alpha_n(J)$ for $J_n(t,r)$ is given by

$$\alpha_n(J)=\frac{T}{2\pi i}\frac{(-1)^n}{n!}p_n(-\log r+\psi(n+1))-\frac{T}{2\pi i}\sum_{k=0}^{n-1}\frac{(-1)^n\,r^{n-k}p_k}{n!\,(n-k)}. \tag{94}$$

Using (34) and setting $r=1$, we have

$$|\alpha_n(J)|=\frac{TM_P(1)}{2\pi}\frac{\psi(n+1)}{a^n}(1+o(1)),n\to\infty. \tag{95}$$

Finally, it follows from (67) that

$$\alpha_n(F_0)=\frac{1}{2\pi i}\frac{1}{n!}\int_{C(r)}e^{-a\zeta}P(\zeta)\frac{d\zeta}{\zeta}, \tag{96}$$

and

$$|\alpha_n(F_0)| \leq \frac{e^{ar} M_P}{n!} \tag{97}$$

So we have proved that the coefficient A_n at $(t+a)^n$ is represented in the form given by (86), where $\alpha_n(I), \alpha_n(J)$, and $\alpha_n(F_0)$ are given by (88), (94), and (96), respectively. Analysis of the estimates for $\alpha_n(I), \alpha_n(J)$, and $\alpha_n(F_0)$ given by (89), (95), and (97) shows that the term (88) gives the main contribution to the asymptotics of A_n as $n \to \infty$. We have the estimae

$$|A_n| = \frac{TM_P(1)}{2\pi} \frac{\sqrt{n+3}}{a^n} (1 + o(1)), n \to \infty \tag{98}$$

which shows that the power series $\sum_{n=0}^{\infty} A_n t^n$ is absolutely convergent in the circle of radius a.

It turns out that we have an alternative proof of Theorem 3 based on Theorem 1. Indeed, combining expressions (82), (83), (87), (93), we have for $|\arg(t+a)| < a$

$$F(t) = -\frac{T}{2\pi i} \sum_{k=0}^{n} \frac{p_k}{r^k} \frac{(-r(t+a))^k}{k!} \log(t+a) + \sum_{n=0}^{n} A_n (t+a)^n + o((t+a)^n), t \to -a, \tag{99}$$

where A_n are given by (86). Since $p_k = F^{(k)}(0)$ we have

$$-\lim_{n\to\infty} \frac{T}{2\pi i} \sum_{k=0}^{n} \frac{p_k}{r^k} \frac{(-r(t+a))^k}{k!} \log(t+a) = -\frac{T}{2\pi i} F(-(t+a)) \log(t+a),$$

so (99) can be rewritten as

$$F(t) = -\frac{T}{2\pi i} F(-(t+a)) \log(t+a) + \sum_{n=0}^{n} A_n (t+a)^n + o((t+a)^n), t \to -a.$$

It remains only to evaluate more accurately the remainder $o((t+a)^n)$ in (87). Let us return to the expression (84) and represent it in the form

$$I_n(t,r) = \frac{T}{2\pi i} \int_r^{+\infty} e^{-(t+a)\zeta} \left(P(\zeta) - \sum_{k=0}^{n} p_k/\zeta^k \right) \frac{d\zeta}{\zeta}$$

which can be rewritten as

$$I_n(t,r) = \frac{T}{2\pi i} \int_r^{+\infty} e^{-(t+a)\zeta} \mathrm{P}_{n+1}(\zeta) \frac{d\zeta}{\zeta}, \tag{100}$$

where $\mathrm{P}_{n+1}(\zeta)$ is given by (30). Using the expansion

$$e^{-(t+a)\zeta} = \sum_{m=0}^{n-1} \frac{(-1)^m (t+a)^m \zeta^m}{m!} + \frac{(-1)^n (t+a)^n \zeta^n}{n!} e^{-(t+a)\zeta(r)},$$

where $r < \zeta(r) < \infty$, and notation (88), the expression (100) can be rewritten as

$$I_n(t,r) = \sum_{k=0}^{n-1} \alpha_k(I)(t+a)^k + R(t,n,r),$$

where

$$R(t,n,r) = \frac{T}{2\pi i}\frac{(-1)^n(t+a)^n}{n!}\int_r^{+\infty} \zeta^n e^{-(t+a)\zeta(r)} \mathrm{P}_{n+1}(\zeta)\frac{d\zeta}{\zeta}. \qquad (101)$$

Applying the estimate for the remainders given by (32) we have for $n = 0, 1, \ldots,$

$$|\mathrm{P}_{n+1}(\zeta)| \le M_P(r)\frac{(n+1)!\sqrt{n+4}}{a^{n+1}\zeta^{n+1}},$$

which together with (101) yields the inequality

$$|R(t,n,r)| \le \frac{TM_P(r)}{2\pi}\frac{(n+1)\sqrt{n+4}}{a^n}|t+a|^n, n = 0, 1, \ldots.$$

Using estimate (86) this completes the alternative proof of Theorem 3. ▲

7 Appendix. The perturbed Whittaker equation

We have replaced the differential equation (11) by a system of functional equations

$$\begin{aligned} P_1\left(\zeta e^{\pi i}\right) &= P_1\left(\zeta e^{-\pi i}\right) + Te^{-a\zeta}P_1(\zeta), \\ P_2\left(\zeta e^{\pi i}\right) &= P_2\left(\zeta e^{-\pi i}\right) + Te^{a\zeta}P_2(\zeta). \end{aligned}$$

We note that these two equations of the system are not linked. A reason for this is that the coefficient $A(\zeta)$ is an even function. However, typically, such equations are intertwined. It is enough to add the term $\frac{b}{\zeta}$ to $A(\zeta)$ to obtain from (11) an intertwined differential equation

$$\frac{d^2u}{d\zeta^2} = \left(\frac{a^2}{4} + \frac{b}{\zeta} + \frac{a_0}{\zeta^2} + \frac{a_1}{\zeta^4} + \ldots\right)u.$$

We consider a more general equation

$$\frac{d^2u}{d\zeta^2} = \left(\frac{a^2}{4} + \frac{b}{\zeta} + \frac{B(\zeta)}{\zeta^2}\right)u, \qquad (102)$$

where $B(\zeta) = \sum_{k=0}^{\infty} b_k/\zeta^k$ is an entire function of $1/\zeta$. A special case $a = 1, b = -\kappa, b_0 = \mu^2 - 1/4, b_k = 0$ for $k = 1, 2, \ldots,$ is known as Whittaker's differential equation. Thus, (102) can be considered as a perturbed Whittaker equation. Let us assume that $a > 0$. Then it can be proved that

there exists a pair of linearly independent solutions $u_1(\zeta)$ and $u_2(\zeta)$ of (102) such that

$$\begin{array}{l} u_1(\zeta) = e^{-\frac{a}{2}\zeta}\zeta^{-\frac{b}{a}}P_1(\zeta), -\frac{3\pi}{2} < \arg\zeta < \frac{3\pi}{2}, \\ u_2(\zeta) = e^{\frac{a}{2}\zeta}\zeta^{\frac{b}{a}}P_2(\zeta), -\frac{\pi}{2} < \arg\zeta < \frac{5\pi}{2} \end{array} \tag{103}$$

where

$$P_1(\zeta), P_2(\zeta) = 1 + o(1) \tag{104}$$

as $\zeta \to \infty$ along any ray of these sectorial regions. It can be shown that the solutions $u_1(\zeta)$ and $u_2(\zeta)$ are uniquely determined by their asymptotics given by (103) and (104) and that $P_1(\zeta)$ and $P_2(\zeta)$ admit analytical continuation along any path not crossing $\zeta = 0$.

Since $u_1\left(\zeta e^{2\pi i}\right)$ and $u_2\left(\zeta e^{2\pi i}\right)$ are also solutions of (102) and since $u_1(\zeta)$ and $u_2\left(\zeta e^{2\pi i}\right)$ are linearly independent solutions we have

$$\begin{array}{l} u_1\left(\zeta e^{2\pi i}\right) = Au_1(\zeta) + Bu_2\left(\zeta e^{2\pi i}\right) \\ u_2\left(\zeta e^{2\pi i}\right) = Cu_2(\zeta) + Du_1(\zeta) \end{array}, \tag{105}$$

where A, B, C, D are complex constants. Using (103), we have

$$\begin{array}{c} u_1\left(\zeta e^{2\pi i}\right) = e^{-2\pi i\frac{b}{a}}e^{-\frac{a}{2}\zeta}\zeta^{-\frac{b}{a}}P_1\left(\zeta e^{2\pi i}\right) \\ u_2\left(\zeta e^{2\pi i}\right) = e^{2\pi i\frac{b}{a}}e^{\frac{a}{2}\zeta}\zeta^{\frac{b}{a}}P_2\left(\zeta e^{2\pi i}\right) \end{array},$$

which allows us to rewrite (105) in the form

$$\begin{array}{l} e^{-2\pi i\frac{b}{a}}e^{-\frac{a}{2}\zeta}\zeta^{-\frac{b}{a}}P_1\left(\zeta e^{2\pi i}\right) = Ae^{-\frac{a}{2}\zeta}\zeta^{-\frac{b}{a}}P_1(\zeta) + Be^{2\pi i\frac{b}{a}}e^{\frac{a}{2}\zeta}\zeta^{\frac{b}{a}}P_2\left(\zeta e^{2\pi i}\right) \\ e^{2\pi i\frac{b}{a}}e^{\frac{a}{2}\zeta}\zeta^{\frac{b}{a}}P_2\left(\zeta e^{2\pi i}\right) = Ce^{\frac{a}{2}\zeta}\zeta^{\frac{b}{a}}P_2(\zeta) + De^{-\frac{a}{2}\zeta}\zeta^{-\frac{b}{a}}P_1(\zeta) \end{array}. \tag{106}$$

Let us simplify (106)

$$e^{-2\pi i\frac{b}{a}}P_1\left(\zeta e^{2\pi i}\right) = AP_1(\zeta) + Be^{2\pi i\frac{b}{a}}e^{a\zeta}\zeta^{\frac{2b}{a}}P_2\left(\zeta e^{2\pi i}\right), \tag{107}$$

$$e^{2\pi i\frac{b}{a}}P_2\left(\zeta e^{2\pi i}\right) = CP_2(\zeta) + De^{-a\zeta}\zeta^{-\frac{2b}{a}}P_1(\zeta). \tag{108}$$

and analyze the last pair of equations. Considering the equation (107) we assume that $\arg\zeta \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$, thus $\arg\left(\zeta e^{2\pi i}\right) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Then letting ζ go to infinity, and taking into account that $P_1(\zeta) \to 1, P_2\left(\zeta e^{2\pi i}\right) \to 1$,and that the term $Be^{a\zeta}\zeta^{\frac{2b}{a}}P_2\left(\zeta e^{2\pi i}\right)$ is exponentially small, it follows that

$$A = e^{-2\pi i\frac{b}{a}}. \tag{109}$$

A similar analysis for (108) and for $\arg\zeta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ shows that

$$C = e^{2\pi i\frac{b}{a}}. \tag{110}$$

Using (109) and (110), the system of equations (107) and (108) can be rewritten as

$$P_1\left(\zeta e^{2\pi i}\right) = P_1(\zeta) + Be^{4\pi i\frac{b}{a}} e^{a\zeta} \zeta^{\frac{2b}{a}} P_2\left(\zeta e^{2\pi i}\right), \tag{111}$$

$$P_2\left(\zeta e^{2\pi i}\right) = P_2(\zeta) + De^{-2\pi i\frac{b}{a}} e^{-a\zeta} \zeta^{-\frac{2b}{a}} P_1(\zeta). \tag{112}$$

Setting

$$T_1 = Be^{4\pi i\frac{b}{a}}, T_2 = De^{-2\pi i\frac{b}{a}}$$

we have finally

$$P_1\left(\zeta e^{2\pi i}\right) = P_1(\zeta) + T_1 e^{a\zeta} \zeta^{\frac{2b}{a}} P_2\left(\zeta e^{2\pi i}\right), \tag{113}$$

$$P_2\left(\zeta e^{2\pi i}\right) = P_2(\zeta) + T_2 e^{-a\zeta} \zeta^{-\frac{2b}{a}} P_1(\zeta). \tag{114}$$

Note that the above pair of relations can also be rewritten as

$$P_1\left(\zeta e^{\pi i}\right) = P_1\left(\zeta e^{-\pi i}\right) + T_1 e^{-2\pi i\frac{b}{a}} e^{-a\zeta} \zeta^{\frac{2b}{a}} P_2\left(\zeta e^{\pi i}\right), \tag{115}$$

$$P_2\left(\zeta e^{\pi i}\right) = P_2\left(\zeta e^{-\pi i}\right) + T_2 e^{2\pi i\frac{b}{a}} e^{a\zeta} \zeta^{-\frac{2b}{a}} P_1\left(\zeta e^{-\pi i}\right). \tag{116}$$

Our principal idea is to consider the system of equations (115) and (116) separately of differential equation. Assuming that $P_1(\zeta)$ and $P_2(\zeta)$ are analytic and bounded in the regions $-\frac{3\pi}{2} < \arg\zeta < \frac{3\pi}{2}$ and $-\frac{\pi}{2} < \arg\zeta < \frac{5\pi}{2}$, respectively and representing these functions in the form

$$P_1(\zeta) = \zeta \int_0^{\infty} e^{-\zeta t} F_1(t)\, dt$$

and

$$P_2(\zeta) = \zeta \int_0^{\infty \cdot e^{\pi i}} e^{-\zeta t} F_2(t)\, dt,$$

we claim the following statement.

Theorem. *(i) Functions $F_1(t)$ and $F_2(t)$ admit analytical continuation to the t-plane punctured at points $t = 0$ and $t = -a$ for $F_1(t)$ and at points $t = 0$ and $t = a$ for $F_2(t)$;*

(ii) there exist branches of $F_1(t)$ and $F_2(t)$ analytic in the t-plane cut along $(-\infty, -a)$ and $(a, +\infty)$, respectively;

(iii) $F_1(t)$ and $F_2(t)$ satisfy a dual system of monodromic relations.

The most nontrivial is the assertion (ii). Setting

$$p_k^{(1)} = F_1^{(k)}(0), p_k^{(2)} = F_2^{(k)}(0), k = 0, 1 \ldots,$$

it follows from this assertion, in particular, that

$$-\frac{3\pi}{2} < \arg\zeta < \frac{3\pi}{2} \Rightarrow \lim_{\zeta\to\infty} P_1(\zeta) = F_1(0),$$

$$-\frac{\pi}{2} < \arg\zeta < \frac{5\pi}{2} \Rightarrow \lim_{\zeta\to\infty} P_2(\zeta) = F_2(0),$$

and moreover that $P_1(\zeta)$ and $P_2(\zeta)$ can be expanded into asymptotic series

$$P_1(\zeta) \sim \sum_{k=0}^{\infty} \frac{p_k^{(1)}}{\zeta^k}, -\frac{3\pi}{2} < \arg\zeta < \frac{3\pi}{2},$$
$$P_2(\zeta) \sim \sum_{k=0}^{\infty} \frac{p_k^{(2)}}{\zeta^k}, -\frac{\pi}{2} < \arg\zeta < \frac{5\pi}{2}.$$

8 Conclusion

Our aim for the future is to extend the approach described here to more general systems of functional monodromic equations that are generated by linear differential equations or systems of differential equations with an irregular singular point of arbitrary Poincaré rank at infinity. The pBde is the simplest differential equation that can be reduced to a single functional equation while preserving most of the difficulties arising in the general case. It is for this reason that in this initial study we have limited our attention to a detailed consideration of the pBde.

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